On the duality between the hyperbolic Sutherland and the rational Ruijsenaars–Schneider models

2009 J. Phys. A: Math. Theor. 42 185202

(http://iopscience.iop.org/1751-8121/42/18/185202)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 185202 (13pp) doi:10.1088/1751-8113/42/18/185202

On the duality between the hyperbolic Sutherland and the rational Ruijsenaars–Schneider models

L Fehér[1,2] **and C Klimčík**[3]

[1] Department of Theoretical Physics, MTA KFKI RMKI H-1525 Budapest, POB 49, Hungary
[2] Department of Theoretical Physics, University of Szeged Tisza Lajos krt 84-86, H-6720 Szeged, Hungary
[3] Institut de mathématiques de Luminy, 163, Avenue de Luminy, 13288 Marseille, France

E-mail: lfeher@rmki.kfki.hu and klimcik@iml.univ-mrs.fr

Received 17 January 2009, in final form 17 March 2009
Published 8 April 2009
Online at stacks.iop.org/JPhysA/42/185202

Abstract

We consider two families of commuting Hamiltonians on the cotangent bundle of the group $GL(n, \mathbb{C})$, and show that upon an appropriate *single* symplectic reduction they descend to the spectral invariants of the hyperbolic Sutherland and of the rational Ruijsenaars–Schneider Lax matrices, respectively. The duality symplectomorphism between these two integrable models that was constructed by Ruijsenaars using direct methods can then be interpreted geometrically simply as a gauge transformation connecting two cross sections of the orbits of the reduction group.

PACS number: 02.30.Ik

1. Introduction

Around 20 years ago, Ruijsenaars [1] undertook a task of constructing action-angle variables for the non-relativistic and relativistic Calogero models of type A_n (for reviews of these models see, e.g., [2–5]). In each case, he made use of a commutation relation satisfied by the Lax matrix of the model under study and another matrix function of the phase-space variables, which he exhibited directly. By conjugating these matrices so as to make the Lax matrix diagonal, he noted that the conjugate of the other matrix becomes the Lax matrix of another Calogero-type model whose particle-position variables are furnished by the eigenvalues of the original Lax matrix, i.e., the action variables of the original model. The so-obtained duality between model 1 and model 2 is thus characterized by the fact that the action variables of model 1 are the particle-position variables of model 2, and vice versa. This observation was used in [1] to derive integration algorithms for the commuting flows and to calculate the scattering data. The simplest manifestation of the duality occurs in the rational Calogero model, which is actually self-dual [1, 6]. The self-duality of this model admits a nice geometric 'explanation' in terms of the symplectic reduction due to Kazhdan, Kostant and Sternberg [7, 2]. As it will serve as a paradigm motivating our considerations, we next outline this explanation in some detail.

In fact, Kazhdan, Kostant and Sternberg reduced the cotangent bundle,

$$T^*u(n) \simeq u(n) \times u(n) = \{(x, y)\}, \tag{1.1}$$

by means of the adjoint action of the group $U(n)$, imposing the moment map constraint

$$[x, y] = \mathrm{i}\kappa(\mathbf{1}_n - ww^\dagger) := \mu_\kappa, \tag{1.2}$$

where all n components of the column vector w are equal to 1, and κ is a real constant. The evaluation functions $X(x, y) := \mathrm{i}x$ and $Y(x, y) := \mathrm{i}y$ can be viewed as 'unreduced Lax matrices' since $\{\mathrm{tr}(X^k)\}$ and $\{\mathrm{tr}(Y^k)\}$ form two Abelian subalgebras in the Poisson algebra $C^\infty(T^*u(n))$. These Abelian algebras survive the reduction, because their elements are $U(n)$ invariants. If one describes the reduced phase space in terms of a gauge slice where X is diagonal, then—by solving the moment map constraint—Y becomes the Lax matrix of the rational Calogero model whose action variables are the eigenvalues of Y. If one describes the reduced phase space in terms of a gauge slice where Y is diagonal, then X becomes the Lax matrix of the 'dual Calogero model'. The correspondence between the variables of the two Calogero models is obviously a symplectomorphism, as it represents the transformation between two gauge slices realizing the *same* reduced phase space. The self-duality stems from the symmetrical roles of x and y, and the commutation relation of the Lax matrices is just the constraint (1.2) in disguise.

Ruijsenaars hinted in [1, 8, 9] that there might exist a similar geometric picture behind the duality in other cases as well, which he left as a problem for 'the aficionados of Lie theory'. Later Gorsky and coworkers [10–12] (see also [13, 14]) introduced interesting new ideas and confirmed this expectation in several cases. In particular, they derived the local version of the so-called $\mathrm{III_b}$ trigonometric Ruijsenaars–Schneider model [3, 9, 15], by reducing a Hamiltonian system on the magnetic cotangent bundle of the loop group of $U(n)$ [10][4]. The investigations in [10–14] focused on the local aspects and did not touch on the quite tricky global definition of the pertinent gauge slices, which is necessary to obtain complete commuting flows. We believe, however, that the reduction approach works also *globally* and that it is possible to characterize the Ruijsenaars duality in a *finite-dimensional* symplectic reduction picture in *all* cases studied in [1, 8, 9]. We plan to explore this issue systematically in a series of papers, and here we report the first results of our analysis.

In this paper, we study a case of the duality which has not been previously described in the symplectic reduction framework. Namely, we expound the geometric picture that links together the dual pair consisting of the hyperbolic Sutherland model and the rational Ruijsenaars–Schneider model [1]. In section 2, we start with two sets of 'canonical integrable systems' on the cotangent bundle of the real Lie group $GL(n, \mathbb{C})$, whose Hamiltonians span two commutative families, $\{H_j\}$ and $\{\hat{H}_k\}$. By 'canonical integrability' we simply mean that one can directly write down the Hamiltonian flows. Then, in section 3, we describe a symplectic reduction of $T^*GL(n, \mathbb{C})$ for which our canonical integrable systems descend to the reduced phase space. By using the shifting trick of symplectic reduction, we exhibit two distinguished cross sections of the orbits of the gauge group that define two models of the reduced phase space. In terms of cross section S_1, the family $\{H_j\}$ translates into the action variables of the Sutherland model and the family $\{\hat{H}_k\}$ becomes equivalent to the Sutherland particle coordinates. Cross section S_1 is described by theorem 1, which summarizes well-known results

[4] It is worth noting that besides the $\mathrm{III_b}$ model there exist also other important, physically different real forms [3, 9] of the complex trigonometric Ruijsenaars–Schneider model.

[2]. In terms of cross section S_2, the family $\{H_j\}$ translates into the coordinate variables of the rational Ruijsenaars–Schneider model and the family $\{\hat{H}_k\}$ gives the action variables of this model. We call cross section S_2 the 'Ruijsenaars gauge slice'. Its characterization by theorem 2 is our principal technical result.

The duality symplectomorphism between the hyperbolic Sutherland and the rational Ruijsenaars–Schneider model will be realized as the gauge transformation between the cross sections S_1 and S_2 mentioned above. In addition, analogously to the case of the rational Calogero model, the symplectic reduction immediately yields integration algorithms for the commuting flows of the dual pair of models, and allows us to recognize the commutation relations of the Lax matrices used by Ruijsenaars as equivalents to the moment map constraint of the reduction. These consequences of theorem 1 and theorem 2 are developed in section 4.

The self-contained presentation of the relatively simple example of the Ruijsenaars duality that follows may also facilitate the geometric understanding of this remarkable phenomenon in more complicated cases.

2. Canonical integrable systems on $T^*GL(n,\mathbb{C})$

Here we describe the two families of canonical integrable systems and their symmetries that will be used to derive the dual pair of integrable many-body models by symplectic reduction. For a general reference on symplectic reduction, we mention the textbook [16].

Consider the real Lie algebra $\mathcal{G} := gl(n,\mathbb{C})$ and equip it with the invariant bilinear form

$$\langle X, Y\rangle := \Re\mathrm{tr}(XY) \qquad \forall\, X, Y \in \mathcal{G}, \tag{2.1}$$

which allows us to identify $\mathcal{G}$ with $\mathcal{G}^*$ by the map $\jmath : \mathcal{G}^* \to \mathcal{G}$ as

$$\langle \jmath(\alpha), X\rangle = \alpha(X) \qquad \forall\, \alpha \in \mathcal{G}^*, \qquad X \in \mathcal{G}. \tag{2.2}$$

Then use left trivialization to obtain a model of the cotangent bundle of the real Lie group $G := GL(n,\mathbb{C})$ as

$$T^*G \simeq G \times \mathcal{G} = \{(g, J^R) \mid g \in G, J^R \in \mathcal{G}\}, \tag{2.3}$$

where $\alpha_g \in T^*_g G$ is represented by $\big(g, \jmath \circ L^*_g(\alpha_g)\big) \in G \times \mathcal{G}$ with the left translation $L_g \in \mathrm{Diff}(G)$. In terms of this model, the canonical symplectic form Ω of T^*G takes the form

$$\Omega = \ \mathrm{d}\langle J^R, g^{-1}\,\mathrm{d}g\rangle. \tag{2.4}$$

Next introduce the matrix functions $\mathcal{L}_1$ and $\mathcal{L}_2$ on T^*G by the definitions

$$\mathcal{L}_1(g, J^R) := J^R \qquad \text{and} \qquad \mathcal{L}_2(g, J^R) := gg^\dagger. \tag{2.5}$$

We may think of these as 'unreduced Lax matrices' since they generate the Hamiltonians

$$H_j := \frac{1}{j}\Re\mathrm{tr}\big(\mathcal{L}_1^j\big), \qquad j = 1, \ldots, n, \tag{2.6}$$

and

$$\hat{H}_k := \frac{1}{2k}\mathrm{tr}\big(\mathcal{L}_2^k\big), \qquad k = \pm 1, \ldots, \pm n, \tag{2.7}$$

so that both $\{H_j\}$ and $\{\hat{H}_k\}$ form commuting sets[5] with respect to the Poisson bracket on the phase space T^*G.

[5] We could have admitted also the imaginary part of the trace in (2.6) and the Hamiltonians in (2.7) are not functionally independent, but these are the definitions that will prove convenient for us.

It is easy to determine the flows of the Hamiltonians introduced above, through any initial value $(g(0), J^R(0))$. In fact, the flow belonging to H_j(2.6) is given by

$$g(t) = g(0)\exp(t(J^R(0))^{j-1}), \qquad J^R(t) = J^R(0). \tag{2.8}$$

The flow generated by $\hat{H}_k$ (2.7) reads as

$$J^R(t) = J^R(0) - t(g^\dagger(0)g(0))^k, \qquad g(t) = g(0). \tag{2.9}$$

We are going to reduce the phase space T^*G by using the symmetry group

$$K := U(n)^L \times U(n)^R, \tag{2.10}$$

where the notation reflects the fact that the two $U(n)$ factors operate by left and right multiplications, respectively. This means that an element $(\eta_L, \eta_R) \in K$ (with $\eta_{L,R} \in U(n)$) acts by the symplectomorphism Ψ_{η_L,η_R} defined by

$$\Psi_{\eta_L,\eta_R}(g, J^R) := \left(\eta_L g \eta_R^{-1}, \eta_R J^R \eta_R^{-1}\right). \tag{2.11}$$

This action is generated by an equivariant moment map. To describe this map, let us note that every $X \in \mathcal{G}$ can be uniquely decomposed (the Cartan decomposition) as

$$X = X_+ + X_- \qquad \text{with} \qquad X_+ \in u(n), \qquad X_- \in \mathrm{i}u(n), \tag{2.12}$$

i.e., into anti-Hermitian and Hermitian parts. Identify $u(n)$ with $u(n)^*$ by the 'scalar product' $\langle . \, , \, . \rangle$ restricted to $u(n) \subset \mathcal{G}$. Then the moment map $\Phi : T^*G \to u(n)^L \oplus u(n)^R$ reads

$$\Phi(g, J^R) = \left((gJ^R g^{-1})_+, -J^R_+\right). \tag{2.13}$$

The Hamiltonians H_j and $\hat{H}_k$ are invariant under the symmetry group K. Hence the commutative character of the families $\{H_j\}$ and $\{\hat{H}_k\}$ survives any symplectic reduction based on this symmetry group. It is also clear that the flows of the reduced Hamiltonians will be provided as projections of the above-given obvious flows to the reduced phase space. However, in general it is a matter of 'art and good luck' to find a value of the moment map that leads to interesting reduced systems. In the present case, it is well known [2] that by setting the $u(n)^R$-component of the moment map Φ to zero, and by setting the $u(n)^L$-component equal to the constant μ_κ defined in (1.2), one obtains the hyperbolic Sutherland model from the Hamiltonian system (T^*G, Ω, H_2).

Our goal is to characterize the reduced Hamiltonian systems coming from (T^*G, Ω, H_j) and from $(T^*G, \Omega, \hat{H}_k)$. To describe the latter systems, it will be technically very convenient to make use of the so-called shifting trick of symplectic reduction. This means that before performing the reduction we extend the phase space by a coadjoint orbit. In the present case, we consider the $U(n)$ orbit through $-\mu_\kappa$ given by

$$\mathcal{O}^L_\kappa := \{\mathrm{i}\kappa(vv^\dagger - \mathbf{1}_n) | v \in \mathbb{C}^n, |v|^2 = n\}. \tag{2.14}$$

The orbit carries its own (Kirillov–Kostant–Souriau) symplectic form, which we denote by $\Omega^{\mathcal{O}}$. The vector v matters only up to the phase and $\left(\mathcal{O}^L_\kappa, \Omega^{\mathcal{O}}\right)$ can be identified as a copy of $\mathbb{C}P_{n-1}$ endowed with a multiple of the Kähler form defined by the Fubini–Study metric.

3. Symplectic reduction of the extended phase space

The extended phase space to consider now is

$$T^*G \times \mathcal{O}^L_\kappa = \{(g, J^R, \xi)\} \tag{3.1}$$

with the symplectic form

$$\Omega^{\text{ext}} = \Omega + \Omega^{\mathcal{O}}. \tag{3.2}$$

The symmetry group K acts by the symplectomorphisms $\Psi^{\text{ext}}_{\eta_L,\eta_R}$ given by

$$\Psi^{\text{ext}}_{\eta_L,\eta_R}(g, J^R, \xi) := \big(\eta_L g \eta_R^{-1}, \eta_R J^R \eta_R^{-1}, \eta_L \xi \eta_L^{-1}\big), \tag{3.3}$$

and the corresponding moment map Φ^{ext} is

$$\Phi^{\text{ext}}(g, J^R, \xi) = \big((g J^R g^{-1})_+ + \xi, -J^R_+\big). \tag{3.4}$$

We are going to reduce at the zero value of the extended moment map, i.e., we wish to describe the reduced phase space

$$T^*G \times \mathcal{O}^L_\kappa //_0 K. \tag{3.5}$$

In our case, this space of K-orbits is a smooth manifold, as will be seen from its models. It is equipped with the reduced symplectic form, Ω^{red}, which is characterized by the equality

$$\pi^* \Omega^{\text{red}} = \iota^* \Omega^{\text{ext}}. \tag{3.6}$$

Here π is the submersion from $(\Phi^{\text{ext}})^{-1}(0)$ to the space of orbits in (3.5), and ι is the injection from $(\Phi^{\text{ext}})^{-1}(0)$ into $T^*G \times \mathcal{O}^L_\kappa$. Before turning to the details, a few remarks are in order.

First, note that one can reduce in steps, initially implementing only the reduction by the factor $U(n)^R$ of K (2.10). This leads to the equality

$$T^*G \times \mathcal{O}^L_\kappa //_0 K = \big[T^*(G/U(n)^R) \times \mathcal{O}^L_\kappa\big] //_0 U(n)^L, \tag{3.7}$$

where $G/U(n)^R$ can be viewed as the symmetric space of positive definite Hermitian matrices. It is for computational convenience that we start from the larger phase space $T^*GL(n, \mathbb{C})$.

Second, the advantage of the shifting trick is that convenient models of the reduced phase space may become available as cross sections of the K-orbits in $(\Phi^{\text{ext}})^{-1}(0)$, which are more difficult to realize without using the auxiliary orbital degrees of freedom. But in principle, one can always do without the shifting trick: in our case we have $T^*G \times \mathcal{O}^L_\kappa //_0 K \equiv T^*G //_{(\mu_\kappa,0)} K$, with the reduced phase space defined by (Marsden–Weinstein) point reduction on the right-hand side of the equality.

Third, if we define

$$\xi(v) := \mathrm{i}\kappa(v v^\dagger - \mathbf{1}_n) \qquad \forall\, v \in \mathbb{C}^n \qquad \text{with} \qquad |v|^2 = n, \tag{3.8}$$

then we have

$$\eta \xi(v) \eta^{-1} = \xi(\eta v) \qquad \forall\, \eta \in U(n). \tag{3.9}$$

This means that the obvious action of $U(n)$ on $\mathbb{C}^n$ underlies the coadjoint action on the orbit $\mathcal{O}^L_\kappa$, and we shall use this to transform the $\mathbb{C}^n$-vectors v that correspond to the elements $\xi(v)$.

Fourth, we define the extended Hamiltonians H^{ext}_j and $\hat{H}^{\text{ext}}_k$ by

$$H^{\text{ext}}_j(g, J^R, \xi) := H_j(g, J^R), \qquad \hat{H}^{\text{ext}}_k(g, J^R, \xi) := \hat{H}_k(g, J^R). \tag{3.10}$$

The flows on $T^*G \times \mathcal{O}^L_\kappa$ are obtained from the flows of the unextended Hamiltonians on T^*G simply by adding the relation $\xi(t) = \xi(0)$. We may also define the matrix functions $\mathcal{L}^{\text{ext}}_1$ and $\mathcal{L}^{\text{ext}}_2$ on $T^*G \times \mathcal{O}^L_\kappa$ by

$$\mathcal{L}^{\text{ext}}_1(g, J^R, \xi) = J^R \qquad \text{and} \qquad \mathcal{L}^{\text{ext}}_2(g, J^R, \xi) = g g^\dagger, \tag{3.11}$$

whereby we can write

$$H^{\text{ext}}_j = \frac{1}{j} \Re \mathrm{tr}\big((\mathcal{L}^{\text{ext}}_1)^j\big) \qquad \text{and} \qquad \hat{H}^{\text{ext}}_k = \frac{1}{2k} \mathrm{tr}\big((\mathcal{L}^{\text{ext}}_2)^k\big). \tag{3.12}$$

Now we turn to the description of two alternative models of the reduced phase space that will be shown to carry a dual pair of integrable many-body models. As a preparation, we associate with any vector $q \in \mathbb{R}^n$ the diagonal matrix

$$\mathbf{q} := \mathrm{diag}(q^1, \ldots, q^n). \tag{3.13}$$

We let $\mathcal{C}$ denote the open domain (Weyl chamber)

$$\mathcal{C} := \{q \in \mathbb{R}^n \,|\, q^1 > q^2 > \cdots > q^n\}, \tag{3.14}$$

and equip

$$T^*\mathcal{C} \simeq \mathcal{C} \times \mathbb{R}^n \tag{3.15}$$

with the Darboux form

$$\Omega_{T^*\mathcal{C}}(q, p) := \sum_k \mathrm{d}p_k \wedge \mathrm{d}q^k. \tag{3.16}$$

3.1. *The Sutherland gauge slice S_1*

Let us define the $\mathrm{i}u(n)$-valued matrix function L_1 on $T^*\mathcal{C}$ (cf (3.15)) by the formula

$$L_1(q, p)_{jk} := p_j\delta_{jk} - \mathrm{i}(1 - \delta_{jk})\frac{\kappa}{\sinh(q^j - q^k)}. \tag{3.17}$$

This is just the standard Lax matrix of the hyperbolic Sutherland model [2, 17, 18]. The following result is well known [2, 7], but for readability we nevertheless present it together with a proof.

Theorem 1. *The manifold S_1 defined by*

$$S_1 := \{(e^{\mathbf{q}}, L_1(q, p), -\mu_\kappa) | (q, p) \in \mathcal{C} \times \mathbb{R}^n\} \tag{3.18}$$

is a global cross section of the K-orbits in the submanifold $(\Phi^{\mathrm{ext}})^{-1}(0)$ of $T^*G \times \mathcal{O}^L_\kappa$. If $\iota_1 : S_1 \to T^*G \times \mathcal{O}^L_\kappa$ is the obvious injection, then in terms of the coordinates q, p on S_1 one has

$$\iota_1^*(\Omega^{\mathrm{ext}}) = \sum_k \mathrm{d}p_k \wedge \mathrm{d}q^k. \tag{3.19}$$

Therefore, the symplectic manifold $(S_1, \sum_k \mathrm{d}p_k \wedge \mathrm{d}q^k) \simeq (T^*\mathcal{C}, \Omega_{T^*\mathcal{C}})$ is a model of the reduced phase space defined by (3.5).

Proof. Our task is to bring every element of $(\Phi^{\mathrm{ext}})^{-1}(0)$ to a unique normal form by the 'gauge transformations' provided by the group K. For this purpose, let us introduce the submanifold $\mathcal{P}$ of $GL(n, \mathbb{C})$ given by the positive definite Hermitian matrices. Recall that the exponential map from $\mathrm{i}u(n)$ to $\mathcal{P}$ is a diffeomorphism. By the polar (Cartan) decomposition, every $g \in GL(n, \mathbb{C})$ can be uniquely written as

$$g = g_-g_+ \qquad \text{with} \qquad g_- \in \mathcal{P}, \qquad g_+ \in U(n). \tag{3.20}$$

It is readily seen that g can be transformed by the K-action into an element for which

$$g_+ = \mathbf{1}_n \qquad \text{and} \qquad g_- = e^{\mathbf{q}} \qquad \text{with} \qquad q \in \mathbb{R}^n, \qquad q^1 \geqslant q^2 \geqslant \cdots \geqslant q^n. \tag{3.21}$$

The moment map constraint requires $J^R_+ = 0$ and, by (3.4), for triples of the form $(e^{\mathbf{q}}, J^R_-, \xi(v))$ we are left with the condition

$$(e^{\mathbf{q}} J^R_- e^{-\mathbf{q}})_+ + \xi(v) = 0. \tag{3.22}$$

This implies that the diagonal entries of the Hermitian matrix J^R_- are arbitrary and the diagonal entries of the anti-Hermitian matrix $\xi(v)$ vanish. By (3.8), it follows from $\xi(v)_{jj} = 0$ that

$$v_j = \mathrm{e}^{\mathrm{i}\theta_j}, \qquad \forall j, \tag{3.23}$$

with some phase factors θ_j. The off-diagonal components of the constraint (3.22) are

$$(J^R_-)_{jk} \sinh(q^j - q^k) + \mathrm{i}\kappa v_j \bar{v}_k = 0, \qquad \forall j \neq k. \tag{3.24}$$

Since $v_j \bar{v}_k \neq 0$, we see from (3.21) and (3.24) that q must belong to the open Weyl chamber $\mathcal{C}$ (3.14). Then the residual gauge transformations permitted by the partial gauge fixing condition (3.21) are given precisely by the maximal torus of $\mathbb{T}_n \subset U(n)$, diagonally embedded into $K = U(n)^L \times U(n)^R$. They operate according to

$$(e^{\mathbf{q}}, J_-^R, v) \mapsto (e^{\mathbf{q}}, \tau J_-^R \tau^{-1}, \tau v) \qquad \forall \tau \in \mathbb{T}_n. \tag{3.25}$$

Hence we can completely fix the residual gauge freedom by transforming the vector v into the representative w, whose components are all equal to 1. At the same time, by (3.24), J_-^R becomes equal to $L_1(q, p)$, where $\mathbf{p}$ is the arbitrary diagonal part of J_-^R. The calculation of $\iota_1^*(\Omega^{\text{ext}})$ as well as the rest of the statements of the theorem is now obvious. □

Remark 1. If we spell out the moment map constraint (3.22) for the solution $J_-^R = L_1$, $v = w$ and also multiply this equation both from the left and from the right by $e^{\mathbf{q}}$, then we obtain

$$[e^{2\mathbf{q}}, L_1(q, p)] + 2\mathrm{i}\kappa((e^{\mathbf{q}}w)(e^{\mathbf{q}}w)^\dagger - e^{2\mathbf{q}}) = 0. \tag{3.26}$$

This is the commutation relation for the Lax matrix (3.17) used in [1].

3.2. The Ruijsenaars gauge slice S_2

In this subsection, we denote the elements of $T^*\mathcal{C} = \mathcal{C} \times \mathbb{R}^n$ as pairs $(\hat{p}, \hat{q})$. We introduce the $\mathcal{P}$-valued matrix function L_2 on $T^*\mathcal{C}$ by the formula

$$L_2(\hat{p}, \hat{q})_{jk} = u_j(\hat{p}, \hat{q}) \left[\frac{2\mathrm{i}\kappa}{2\mathrm{i}\kappa + (\hat{p}^j - \hat{p}^k)} \right] u_k(\hat{p}, \hat{q}), \tag{3.27}$$

with the $\mathbb{R}_+$-valued functions

$$u_j(\hat{p}, \hat{q}) := \mathrm{e}^{-\hat{q}_j} \prod_{m \neq j} \left[1 + \frac{4\kappa^2}{(\hat{p}^j - \hat{p}^m)^2} \right]^{\frac{1}{4}}. \tag{3.28}$$

One can calculate the principal minors of L_2 with the help of the Cauchy determinant formula, and thereby confirm that L_2 is indeed a positive definite matrix. Therefore, it admits a unique positive definite square root, and we use it to define the $\mathbb{R}^n$-valued function,

$$v(\hat{p}, \hat{q}) := L_2(\hat{p}, \hat{q})^{-\frac{1}{2}} u(\hat{p}, \hat{q}), \tag{3.29}$$

where $u = (u_1, \ldots, u_n)^T$. It can be verified directly, and can be seen also from the proof below, that $|v(\hat{p}, \hat{q})|^2 = n$. Thus, by using (3.8), we have the $\mathcal{O}_\kappa^L$-valued function

$$\xi(\hat{p}, \hat{q}) := \xi(v(\hat{p}, \hat{q})). \tag{3.30}$$

The function L_2 (3.27) is nothing but the standard Lax matrix of the rational Ruijsenaars–Schneider model [19], with variables denoted by somewhat unusual letters. Using the above notations, we now state the main technical result of the present paper.

Theorem 2. *The manifold S_2 defined by*

$$S_2 := \left\{ (L_2(\hat{p}, \hat{q})^{\frac{1}{2}}, \hat{\mathbf{p}}, \xi(\hat{p}, \hat{q})) | (\hat{p}, \hat{q}) \in \mathcal{C} \times \mathbb{R}^n \right\} \tag{3.31}$$

is a global cross section of the K-orbits in the submanifold $(\Phi^{\text{ext}})^{-1}(0)$ of $T^*G \times \mathcal{O}_\kappa^L$. If $\iota_2 : S_2 \to T^*G \times \mathcal{O}_\kappa^L$ is the obvious injection, then in terms of the coordinates $\hat{p}, \hat{q}$ on S_2 one has

$$\iota_2^*(\Omega^{\text{ext}}) = \sum_k \mathrm{d}\hat{q}_k \wedge \mathrm{d}\hat{p}^k. \tag{3.32}$$

Therefore, the symplectic manifold $(S_2, \sum_k \mathrm{d}\hat{q}_k \wedge \mathrm{d}\hat{p}^k) \simeq (T^*\mathcal{C}, \Omega_{T^*\mathcal{C}})$ is a model of the reduced phase space defined by (3.5).

Proof. Let us begin by noting that by means of the K-action we can transform each element of $(\Phi^{\rm ext})^{-1}(0)$ into an element $(g, J^R, \xi(v))$ that satisfies

$$g = g_- \in \mathcal{P} \qquad \text{and} \qquad J^R = \hat{\mathbf{p}} \qquad \text{with} \qquad \hat{p} \in \mathbb{R}^n, \qquad \hat{p}^1 \geqslant \hat{p}^2 \geqslant \cdots \geqslant \hat{p}^n. \tag{3.33}$$

After this partial gauge fixing the moment map constraint becomes

$$g_-^{-1}\hat{\mathbf{p}}g_- - g_-\hat{\mathbf{p}}g_-^{-1} = 2\xi(v). \tag{3.34}$$

In order to solve this equation, we multiply it both from the left and from the right by g_-, which gives

$$[\hat{\mathbf{p}}, g_-^2] = 2\mathrm{i}\kappa(uu^\dagger - g_-^2), \tag{3.35}$$

where we have combined the unknowns g_- and v to define

$$u := g_- v. \tag{3.36}$$

This equation then permits us to express g_-^2 in terms of $\hat{\mathbf{p}}$ and u as

$$(g_-^2)_{jk} = u_j \left[\frac{2\mathrm{i}\kappa}{2\mathrm{i}\kappa + (\hat{p}^j - \hat{p}^k)}\right]\bar{u}_k. \tag{3.37}$$

By calculating the determinant from the last relation, we obtain

$$\det(g_-^2) = \left(\prod_m |u_m|^2\right)\prod_{j<k}\frac{(\hat{p}^j - \hat{p}^k)^2}{(\hat{p}^j - \hat{p}^k)^2 + 4\kappa^2}. \tag{3.38}$$

Since $\det(g_-^2) \neq 0$, we must have

$$\hat{p}^1 > \hat{p}^2 > \cdots > \hat{p}^n \qquad \text{and} \qquad u_j \neq 0 \quad \forall\, j. \tag{3.39}$$

In particular, $\hat{p}$ must belong to the open Weyl chamber $\mathcal{C}$. This implies that the residual gauge transformations permitted by our partial gauge fixing are generated by the maximal torus $\mathbb{T}_n$ of $U(n)$, diagonally embedded into K, which act according to

$$(g_-, \hat{\mathbf{p}}, v) \mapsto (\tau g_- \tau^{-1}, \hat{\mathbf{p}}, \tau v) \qquad \forall\, \tau \in \mathbb{T}_n. \tag{3.40}$$

By applying these gauge transformations to u in (3.36) we have

$$\tau : u \mapsto \tau u. \tag{3.41}$$

It follows that we obtain a complete gauge fixing by imposing the conditions

$$u_j > 0 \qquad \forall\, j. \tag{3.42}$$

For fixed $\hat{p}$, the positive vector u remains arbitrary, and thus it can be uniquely parametrized by introducing a new variable $\hat{q} \in \mathbb{R}^n$ via equation (3.28).

The outcome of the above discussion is that the manifold S_2 given by (3.31) is a global cross section of the K-orbits in $(\Phi^{\rm ext})^{-1}(0)$, which provides a model of the reduced phase space (3.5). Indeed, formula (3.37) of g_-^2 becomes identical to formula (3.27) of L_2 if we take into account the gauge fixing conditions (3.42) and the parametrization (3.28). At the same time, the inversion of relation (3.36) yields $v(\hat{p}, \hat{q})$ in (3.29).

It remains to show that the variables $(\hat{p}, \hat{q})$ that parametrize S_2 are Darboux coordinates on the reduced phase space. Direct substitution into the symplectic form $\Omega^{\rm ext}$ (3.2) is now cumbersome, since we do not have $L_2(\hat{p}, \hat{q})^{\frac{1}{2}}$ explicitly. We circumvent this problem by proceeding as follows. We define the smooth functions F^a and E^a on $T^*G \times \mathcal{O}_\kappa^L$ by

$$F^a(g, J^R, \xi) := \frac{1}{a}\mathrm{tr}[(J_-^R)^a] \qquad \text{and} \qquad E^a(g, J^R, \xi) := \frac{1}{a}\mathrm{tr}[(gg^\dagger)^a], \qquad a = 1, \ldots, n. \tag{3.43}$$

We also define the functions $\mathcal{F}^a$ and $\mathcal{E}^a$ on the gauge slice S_2 by

$$\mathcal{F}^a(\hat{p},\hat{q}) := \frac{1}{a}\mathrm{tr}[\hat{\mathbf{p}}^a] \qquad \text{and} \qquad \mathcal{E}^a(\hat{p},\hat{q}) := \frac{1}{a}\mathrm{tr}[L_2(\hat{p},\hat{q})^a], \qquad a = 1,\ldots,n. \tag{3.44}$$

It is clear that F^a and E^a are K-invariant functions, and by means of the injection map $\iota_2 : S_2 \to T^*G \times \mathcal{O}^L_\kappa$ we have

$$\iota_2^* F^a = \mathcal{F}^a, \qquad \iota_2^* E^a = \mathcal{E}^a \qquad \forall\, a = 1,\ldots,n. \tag{3.45}$$

Therefore, we can determine the induced Poisson brackets of these functions by two methods. First, denote the Poisson bracket on the extended phase space by $\{.,.\}^{\mathrm{ext}}$. The Poisson bracket of K-invariant functions is again K-invariant and a straightforward calculation gives (for any $1 \leqslant a, b \leqslant n$)

$$\iota_2^*\{E^a, E^b\}^{\mathrm{ext}} = \iota_2^*\{F^a, F^b\}^{\mathrm{ext}} = 0, \qquad \iota_2^*\{E^a, F^b\}^{\mathrm{ext}} = 2\mathrm{tr}\big[(\hat{\mathbf{p}})^{b-1} L_2^a\big]. \tag{3.46}$$

Second, denote by $\{.,.\}^{\mathrm{red}}$ the Poisson bracket on $C^\infty(S_2)$ induced by the reduced symplectic structure. We wish to show that

$$\{\hat{p}^j, \hat{p}^k\}^{\mathrm{red}} = \{\hat{q}_j, \hat{q}_k\}^{\mathrm{red}} = 0 \qquad \text{and} \qquad \{\hat{p}^j, \hat{q}_k\}^{\mathrm{red}} = \delta^j_k. \tag{3.47}$$

Now, if we assume that the last relation holds, then it can be verified by direct calculation[6] (for any $1 \leqslant a, b \leqslant n$) that

$$\{\mathcal{E}^a, \mathcal{E}^b\}^{\mathrm{red}} = \{\mathcal{F}^a, \mathcal{F}^b\}^{\mathrm{red}} = 0, \qquad \{\mathcal{E}^a, \mathcal{F}^b\}^{\mathrm{red}} = 2\mathrm{tr}\big[(\hat{\mathbf{p}})^{b-1} L_2^a\big]. \tag{3.48}$$

From general principles, the restriction of the Poisson bracket of K-invariant functions to a gauge slice always yields the induced Poisson bracket of the restricted functions. By taking into account (3.45), we conclude from the comparison of equations (3.46) and (3.48) that the Poisson bracket on $C^\infty(S_2)$ that arises from the symplectic reduction coincides with the Poisson bracket given in coordinates by (3.47), at least if we restrict our attention to the collection of the functions $\mathcal{F}^a, \mathcal{E}^a$. Thus it remains to prove that the functions $\mathcal{F}^a, \mathcal{E}^a$ can serve as local coordinates around any point from a dense open submanifold of S_2. Indeed, the symplectic form on this dense submanifold could then be written as $\sum_k \mathrm{d}\hat{q}_k \wedge \mathrm{d}\hat{p}^k$. The smoothness of the symplectic form would then allow us to conclude the same on the whole of S_2.

The map from $\mathcal{C} \times \mathbb{R}^n$ to $\mathcal{C} \times (\mathbb{R}_+)^n$ given by $(\hat{p},\hat{q}) \mapsto (\hat{p}, u(\hat{p},\hat{q}))$ is clearly a diffeomorphism, and thus we can use $\hat{p}^k, u_k$ $(k = 1,\ldots,n)$ as coordinates on S_2. When expressed in these new coordinates, we denote our functions of interest as $\tilde{\mathcal{F}}^a$, $\tilde{\mathcal{E}}^a$ and $\tilde{L}_2$:

$$\tilde{\mathcal{F}}^a(\hat{p},u) = \mathcal{F}^a(\hat{p},\hat{q}), \qquad \tilde{\mathcal{E}}^a(\hat{p},u) = \mathcal{E}^a(\hat{p},\hat{q}), \qquad \tilde{L}_2(\hat{p},u) = L_2(\hat{p},\hat{q}) \tag{3.49}$$

if $u = u(\hat{p},\hat{q})$. To finish the proof, it is sufficient to show that the Jacobian determinant of the map $\mathcal{C} \times (\mathbb{R}_+)^n \to \mathbb{R}^n \times \mathbb{R}^n$ given in coordinates by $\hat{p}^k, u_k \mapsto \tilde{\mathcal{F}}^a(\hat{p},u), \tilde{\mathcal{E}}^a(\hat{p},u)$ is non-vanishing on a dense open subset of $\mathcal{C} \times (\mathbb{R}_+)^n$. For this, note from (3.27) and (3.44) that $\tilde{\mathcal{F}}^a$ and $\tilde{\mathcal{E}}^a$ are rational functions of the variables $\hat{p}^k, u_k$, and thus the Jacobian,

$$\det \frac{\partial(\tilde{\mathcal{F}}^a, \tilde{\mathcal{E}}^b)}{\partial(\hat{p}^j, u_k)}, \tag{3.50}$$

is also a rational function of the same variables (that is, the quotient of two polynomials in the $2n$-variables $\hat{p}^k, u_k$). This implies that the Jacobian (3.50) either vanishes identically or is non-vanishing on a dense open subset of $\mathcal{C} \times (\mathbb{R}_+)^n \simeq S_2$.

6 Only the relation $\{\mathcal{E}^a, \mathcal{E}^b\}^{\mathrm{red}} = 0$ requires non-trivial effort, but this was established in [19] as well as in the more recent papers dealing with the dynamical r-matrix structure of the Ruijsenaars–Schneider models [20].

Now we show that the Jacobian (3.50) does not vanish identically. First of all, from the fact that $\tilde{\mathcal{F}}^a$ does not depend on u_k, we see easily that

$$\det \frac{\partial(\tilde{\mathcal{F}}^a, \tilde{\mathcal{E}}^b)}{\partial(\hat{p}^j, u_k)} = \det\left[\frac{\partial \tilde{\mathcal{F}}^a}{\partial \hat{p}^j}\right] \det\left[\frac{\partial \tilde{\mathcal{E}}^b}{\partial u_k}\right]. \tag{3.51}$$

The first determinant on the rhs is the Vandermonde one:

$$\det\left[\frac{\partial \tilde{\mathcal{F}}^a}{\partial \hat{p}^j}\right] = \prod_{i<m} (\hat{p}^m - \hat{p}^i), \tag{3.52}$$

which never vanishes on S_2 due to (3.39). The second determinant on the rhs of (3.51) parametrically depends on $\hat{p}$. In particular, if we take a real parameter $s > 0$ and consider the curve $\hat{p}(s)$ defined by $\hat{p}^j(s) := \mathrm{e}^{(n+1-j)s}$ for $j = 1, \ldots, n$, then we observe that in the limit $s \to \infty$ the matrix $\tilde{L}_2$ (3.49) becomes diagonal:

$$\lim_{s\to\infty} \tilde{L}_2(\hat{p}(s), u) = \mathrm{diag}\big(u_1^2, \ldots, u_n^2\big). \tag{3.53}$$

Hence, in the limit $s \to \infty$, we encounter again a Vandermonde determinant:

$$\lim_{s\to\infty} \det\left[\frac{\partial \tilde{\mathcal{E}}^b}{\partial u_k}\right](\hat{p}(s), u) = 2^n \left(\prod_j u_j\right) \prod_{i<m} \big(u_m^2 - u_i^2\big). \tag{3.54}$$

Obviously, we can choose u in such a way that the rhs of equation (3.54) does not vanish. Then the Jacobian determinant (3.50) does not vanish at $(\hat{p}(s), u)$ for large enough s. □

Remark 2. The consequence (3.35) of the moment map constraint yields the commutation relation satisfied by the Lax matrix L_2 (3.27) and $u(\hat{p}, \hat{q})$ (3.28):

$$[L_2(\hat{p}, \hat{q}), \hat{\mathbf{p}}] + 2\mathrm{i}\kappa\,(u(\hat{p}, \hat{q})u(\hat{p}, \hat{q})^\dagger - L_2(\hat{p}, \hat{q})) = 0, \tag{3.55}$$

which played a crucial role in the analysis in [1].

4. The dual pair of many-body models

We now enumerate important consequences of the results presented in the preceding sections.

(i) Since S_1 (3.18) and S_2 (3.31) are two models of the same reduced phase space (3.5), there exists a *natural symplectomorphism* between these two models:

$$\left(S_1, \sum_k \mathrm{d}p_k \wedge \mathrm{d}q^k\right) \equiv \big(T^*G \times \mathcal{O}^L_\kappa //_0 K, \Omega^{\mathrm{red}}\big) \equiv \left(S_2, \sum_k \mathrm{d}\hat{q}_k \wedge \mathrm{d}\hat{p}^k\right). \tag{4.1}$$

By definition, a point of S_1 is related to that point of S_2 which represents the same element of the reduced phase space.

(ii) The pullbacks of the 'unreduced Lax matrices' (3.11) to S_1 and S_2 satisfy, respectively,

$$\iota_1^* \mathcal{L}_1^{\mathrm{ext}} = L_1 \qquad \text{and} \qquad \iota_2^* \mathcal{L}_2^{\mathrm{ext}} = L_2. \tag{4.2}$$

By the symplectic reduction, the K-invariant Hamiltonians (3.12) descend to the families of Hamiltonians defined on $(S_1, \sum_k \mathrm{d}p_k \wedge \mathrm{d}q^k)$ and on $(S_2, \sum_k \mathrm{d}\hat{q}_k \wedge \mathrm{d}\hat{p}^k)$, respectively, by

$$H_j^{\mathrm{red}} = \frac{1}{j}\mathrm{tr}\big(L_1^j\big), \qquad (j = 1, \ldots, n) \qquad \text{and}$$
$$\hat{H}_k^{\mathrm{red}} = \frac{1}{2k}\mathrm{tr}\big(L_2^k\big), \qquad (k = \pm 1, \ldots, \pm n). \tag{4.3}$$

The commutativity of $\{H_j^{\mathrm{red}}\}$ and $\{\hat{H}_k^{\mathrm{red}}\}$ (where the elements with fixed sign of k form a complete set) is inherited from the commutativity of the unreduced Hamiltonians in (3.12). (Note that $\mathrm{tr}(L_1^j) = \Re\mathrm{tr}(L_1^j)$ since L_1 (3.17) is a Hermitian matrix.)

(iii) According to classical results, L_1 (3.17) is the Lax matrix of the hyperbolic Sutherland model, and L_2 (3.27) is the Lax matrix of the rational Ruijsenaars–Schneider model. The basic Hamiltonians of these many-body models are indeed reproduced as

$$H_{\mathrm{hyp-Suth}}(q,p) \equiv \frac{1}{2}\sum_k p_k^2 + \frac{\kappa^2}{2}\sum_{j\neq k}\frac{1}{\sinh^2(q^j-q^k)} = \frac{1}{2}\mathrm{tr}(L_1(q,p)^2), \tag{4.4}$$

$$H_{\mathrm{rat-RS}}(\hat{p},\hat{q}) \equiv \sum_k \cosh(\hat{q}_k)\prod_{j\neq k}\left[1+\frac{4\kappa^2}{(\hat{p}^k-\hat{p}^j)^2}\right]^{\frac{1}{2}} = \frac{1}{2}\mathrm{tr}(L_2(\hat{p},\hat{q})+L_2(\hat{p},\hat{q})^{-1}). \tag{4.5}$$

The Lax matrices themselves arose naturally (4.2) by means of the symplectic reduction.

(iv) Consider two points of S_1 and S_2 that are related by the symplectomorphism (4.1). Suppose that these two points are parametrized by $(q,p)\in\mathcal{C}\times\mathbb{R}^n$ and by $(\hat{p},\hat{q})\in\mathcal{C}\times\mathbb{R}^n$ according to (3.18) and (3.31), respectively. The fact that they lie on the same K-orbit means, since g_+ in $g = g_-g_+$ (3.20) is fixed to $\mathbf{1}_n$ in both gauges, that there exists some $\eta\in U(n)$ (actually unique up to the center of $U(n)$ that acts trivially) for which

$$\left(\eta e^{\mathbf{q}}\eta^{-1}, \eta L_1(q,p)\eta^{-1}, -\eta\mu_\kappa\eta^{-1}\right) = (L_2(\hat{p},\hat{q})^{\frac{1}{2}}, \hat{\mathbf{p}}, \xi(\hat{p},\hat{q})). \tag{4.6}$$

This shows that the matrix $\hat{\mathbf{p}}$, which encodes the coordinate variables of the rational Ruijsenaars–Schneider model, results by diagonalizing the Sutherland Lax matrix $L_1(q,p)$. The same formula shows that $e^{2\mathbf{q}}$, which encodes the coordinate variables of the hyperbolic Sutherland model, results by diagonalizing the Ruijsenaars–Schneider Lax matrix $L_2(\hat{p},\hat{q})$. The original, direct construction [1] of the map between the phase spaces of the two many-body models relied on diagonalization of the Lax matrices, but in that approach it was quite difficult to prove the canonicity of the map, which comes for free in the symplectic reduction framework.

(v) It is obvious from the above observations that the two many-body models characterized by the Hamiltonians (4.4) and (4.5) are dual to each other in the sense described in the Introduction. On the one hand, the Ruijsenaars–Schneider particle coordinates $\hat{p}^1,\ldots,\hat{p}^n$ *regarded as functions on* S_1 define action variables for the hyperbolic Sutherland model. On the other, the Sutherland particle coordinates $q^1,\ldots,q^n$ *regarded as functions on* S_2 can serve as action variables for the rational Ruijsenaars–Schneider model.

(vi) The well-known solution algorithms [2, 19] for the commuting Hamiltonians exhibited in (4.3) can be viewed as byproducts of the geometric approach. First of all, it should be noted that all the flows generated by the reduced Hamiltonians are *complete*, since this is true for the unreduced Hamiltonians whose flows stay in $(\Phi^{\mathrm{ext}})^{-1}(0)$. By taking an initial value on the gauge slice S_1 and projecting the flow (2.8) back to S_1 we obtain that the reduced Hamiltonian H_j^{red} (4.3) generates the following evolution for the Sutherland coordinate-variables:

$$e^{2\mathbf{q}(t)} = \mathcal{D}\left[e^{\mathbf{q}(0)}\exp\left(2tL_1(0)^{j-1}\right)e^{\mathbf{q}(0)}\right], \tag{4.7}$$

where the zero argument refers to the $t=0$ initial value, and $\mathcal{D}$ denotes the operator that brings its Hermitian matrix argument to diagonal form with eigenvalues in non-increasing order. Similarly, we obtain from (2.9) that the reduced Hamiltonian $\hat{H}_k^{\mathrm{red}}$ (4.3) generates the following flow for the Ruijsenaars–Schneider coordinate variables:

$$\hat{\mathbf{p}}(t) = \mathcal{D}\big[\hat{\mathbf{p}}(0) - tL_2(0)^k\big]. \tag{4.8}$$

We here used that $(g^\dagger(0)g(0))^k = L_2(0)^k$ for any initial value on S_2.

To summarize, we presented the duality between the hyperbolic Sutherland model (4.4) and the rational Ruijsenaars–Schneider model (4.5) in the framework of symplectic reduction. In this way, we obtained a Lie theoretic understanding of results due to Ruijsenaars [1], who originally discovered and investigated the duality by direct means. Our approach also simplifies a considerable portion of the original technical arguments. The general line of reasoning that we followed may be adapted to explore more complicated cases of the duality in the future, too. For example, it will be demonstrated in [21, 22] that the reduction approach works in a conceptually very similar manner for the dualities concerning trigonometric Ruijsenaars–Schneider models.

Finally, let us mention that the dual pairs of models studied by Ruijsenaars at the classical level [1, 8, 9] are associated at the quantum-mechanical level with so-called bispectral problems [23], as was first conjectured in [3] and later confirmed in several papers. Concerning the bispectral property, and in particular the bispectral interpretation of the duality between the hyperbolic Sutherland and the rational Ruijsenaars–Schneider models, the reader may consult [24, 25] and references therein. We expect that this intriguing phenomenon could be understood also in terms of a quantum Hamiltonian reduction counterpart of our approach.

Acknowledgments

LF was partially supported by the Hungarian Scientific Research Fund (OTKA grant T049495) and by the EU network 'ENIGMA' (MRTN-CT-2004-5652). He wishes to thank I Marshall and V Rubtsov for useful discussions and remarks.

References

[1] Ruijsenaars S N M 1988 Action-angle maps and scattering theory for some finite-dimensional integrable systems: I. The pure soliton case *Commun. Math. Phys.* **115** 127–65

[2] Olshanetsky M A and Perelomov A M 1981 Classical integrable finite-dimensional systems related to Lie algebras *Phys. Rep.* **71** 313–400

[3] Ruijsenaars S N M 1990 Finite-dimensional soliton systems *Integrable and Superintegrable Systems* ed B Kupershmidt (Singapore: World Scientific) pp 165–206

[4] Calogero F 2001 *Classical Many-Body Problems Amenable to Exact Treatments* (Berlin: Springer)

[5] Sutherland B 2004 *Beautiful Models* (Singapore: World Scientific)

[6] Airault H, McKean H P and Moser J 1977 Rational and elliptic solutions of the Korteweg–de Vries equation and a related many-body problem *Commun. Pure Appl. Math.* **30** 95–148

[7] Kazhdan D, Kostant B and Sternberg S 1978 Hamiltonian group actions and dynamical systems of Calogero type *Commun. Pure Appl. Math.* **31** 481–507

[8] Ruijsenaars S N M 1994 Action-angle maps and scattering theory for some finite-dimensional integrable systems: II. Solitons, antisolitons and their bound states *Publ. RIMS* **30** 865–1008

[9] Ruijsenaars S N M 1995 Action-angle maps and scattering theory for some finite-dimensional integrable systems: III. Sutherland type systems and their duals *Publ. RIMS* **31** 247–353

[10] Gorsky A and Nekrasov N 1995 Relativistic Calogero–Moser model as gauged WZW theory *Nucl. Phys.* B **436** 582–608

[11] Nekrasov N 1999 Infinite-dimensional algebras, many-body systems and gauge theories *Moscow Seminar in Mathematical Physics* ed A Yu Morozov and M A Olshanetsky (Providence, RI: American Mathematical Society) pp 263–99

[12] Fock V, Gorsky A, Nekrasov N and Rubtsov V 2000 Duality in integrable systems and gauge theories *J. High Energy Phys.* JHEP07(2000)028

[13] Arutyunov G E and Frolov S A 1998 On Hamiltonian structure of the spin Ruijsenaars–Schneider model *J. Phys. A: Math. Gen.* **31** 4203–16

[14] Fock V V and Rosly A A 1999 Poisson structure on moduli of flat connections on Riemann surfaces and the *r*-matrix *Moscow Seminar in Mathematical Physics* ed A Yu Morozov and M A Olshanetsky (Providence, RI: American Mathematical Society) pp 67–86

[15] van Diejen J F and Vinet L 1998 The quantum dynamics of the compactified trigonometric Ruijsenaars–Schneider model *Commun. Math. Phys.* **197** 33–74

[16] Libermann P and Marle C-M 1986 *Symplectic Geometry and Analytical Mechanics* (Dordrecht: Reidel)

[17] Moser J 1975 Three integrable Hamiltonian systems connected with isospectral deformations *Adv. Math.* **16** 197–220

[18] Calogero F, Ragnisco O and Marchioro C 1975 Exact solution of the classical and quantal one-dimensional many-body problems with the two-body potential $V_a(x) = g^2a^2/\sinh^2(ax)$ *Lett. Nuovo Cimento* **13** 383–7

[19] Ruijsenaars S N M and Schneider H 1986 A new class of integrable models and their relation to solitons *Ann. Phys., NY* **170** 370–405

[20] Avan J 2000 Classical dynamical *r*-matrices for Calogero–Moser systems and their genealizations *Calogero–Moser–Sutherland Models* ed J F van Diejen and L Vinet (Berlin: Springer) pp 1–21 and references therein

[21] Fehér L and Klimčík C 2009 Poisson–Lie generalization of the Kazhdan–Kostant–Sternberg reduction *Lett. Math. Phys.* **87** 125–38

[22] Fehér L and Klimčík C Poisson–Lie interpretation of a case of the Ruijsenaars duality (in preparation)

[23] Duistermaat J J and Grünbaum F A 1986 Differential equations in the spectral parameter *Commun. Math. Phys.* **103** 177–240

[24] Chalykh O A 2000 Bispectrality for the quantum Ruijsenaars model and its integrable deformation *J. Math. Phys.* **41** 5139–67

[25] Haine L 2007 KP trigonometric solitons and an adelic flag manifold *SIGMA* **3** 015